Noise robustness and threshold of many-body quantum magic

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Outline

- Background and Preliminaries
- Noise robustness and threshold of magic
- Result 1. Magic induced by broader interactions is less robust
- Result 2. Local magic and magic threshold
- Outlook

Background

- Why quantum computing is potentially more powerful than classical computing?
  - “Magic”, a source of quantum computational advantages

- Noise can significantly undermine the resource features of quantum systems.
  - Noisy intermediate-scale quantum (NISQ) technologies
  - Quantum error correction
  - Has been investigated for features like computational supremacy, entanglement …

Aharonov D, Gao X, Landau Z, et al. A polynomial-time classical algorithm for noisy random circuit sampling[C]//STOC. 2023: 945-957.

Contreras-Tejada P, Palazuelos C, de Vicente J I. Asymptotic survival of genuine multipartite entanglement in noisy quantum networks depends on the topology[J]. PRL, 2022, 128(22): 220501.

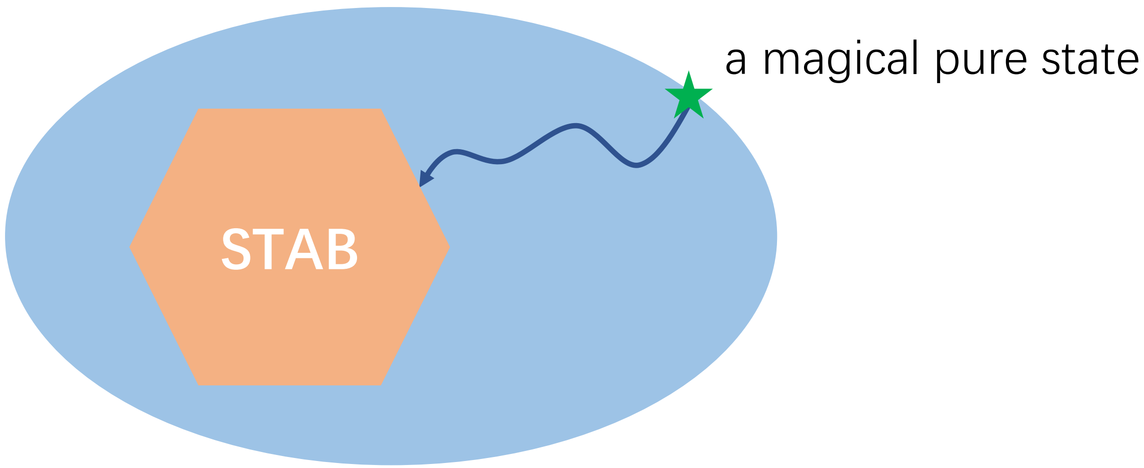
Background

- How noise effects impact magic?

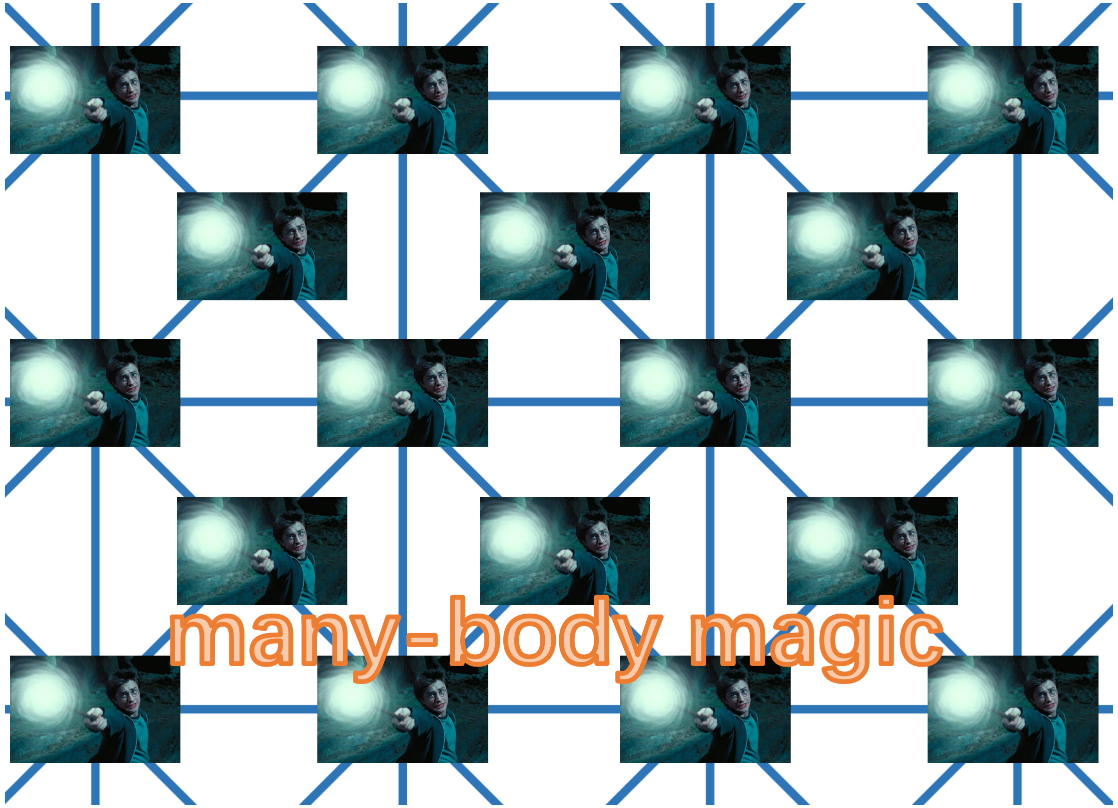
Background

- How noise effects impact magic?

- Noise drives pure states to mixed ones, and as it intensifies, the magic gradually decays and eventually vanishes at a certain point as the state is brought inside the stabilizer hull.



- How noise effects impact magic in large systems with different entanglement structures?
- Generates insights into
  - Interplay between magic and entanglement
  - Design of circuits
  - …



Preliminaries

- A neat yet highly versatile model for entangled magic states: hypergraph states.
  - many-body physics
  - measurement-based quantum computing (MBQC)

- Also provide an apt playground for concretely investigating the relation between entanglement structures and magic properties

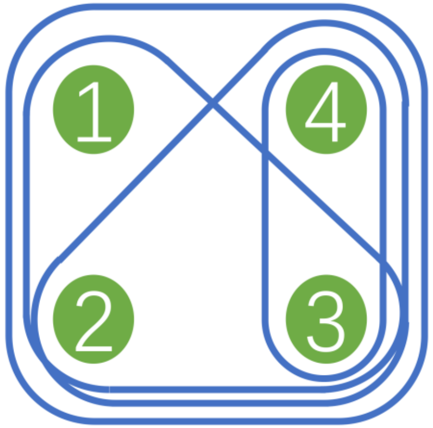
Levin M, Gu Z C. Braiding statistics approach to symmetry-protected topological phases[J]. PRB, 2012, 86(11): 115109.
Miller J, Miyake A. Latent computational complexity of symmetry-protected topological order with fractional symmetry[J]. PRL, 2018, 120(17): 170503.

Raussendorf R, Briegel H J. A one-way quantum computer[J]. PRL, 2001, 86(22): 5188.
Miller J, Miyake A. Hierarchy of universal entanglement in 2D measurement-based quantum computation[J]. npj Quantum Information, 2016, 2(1): 1-6.

Liu Z W, Winter A. Many-body quantum magic[J]. PRX Quantum, 2022, 3(2): 020333.

Preliminaries

- A neat yet highly versatile model for entangled magic states: hypergraph states.

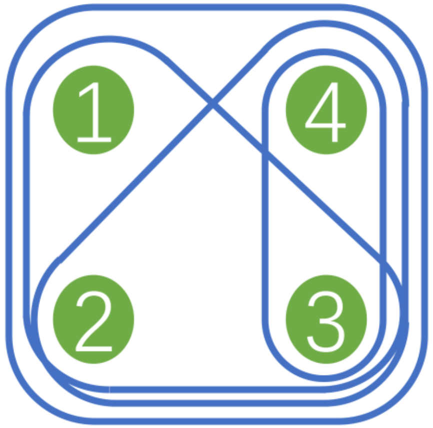


$$E = \{\{3,4\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}$$

$$|\Psi_G\rangle = \mathrm{C}Z_{34}\mathrm{CC}Z_{123}\mathrm{CC}Z_{234}\mathrm{CCC}Z_{1234}|+^4\rangle$$

Preliminaries

- A neat yet highly versatile model for entangled magic states: hypergraph states.



Let $C^{n-1}Z = \mathrm{diag}(1, \cdots, 1, -1)$ denote the multi-controlled-Z gate on n-qubits, with $C^0Z = Z$.

Given a hypergraph $G = \{[n], E\}$, where $[n] := \{1, \cdots, n\}$ is the set of vertices and $E \subset 2^{[n]}$ is the set of hyperedges, an associated hypergraph state is defined by

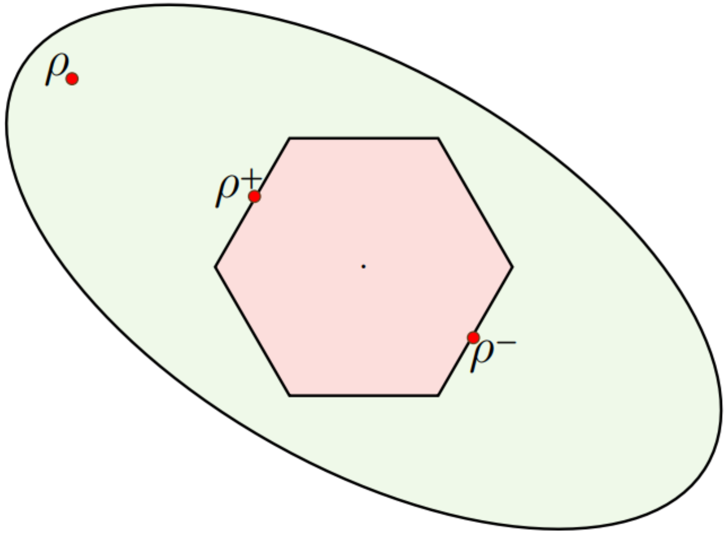
$$|\Psi_G\rangle = \prod_{e \in E} C^{|e|-1} Z_e |+^n\rangle,$$

where $C^{|e|-1}Z_e$ is the multi-controlled-Z gate applied to the qubits in $e \subset [n]$.

The number of vertices contained in an edge e is referred to as the *degree* of the edge.

Preliminaries

- Magic measure for mixed states: robustness of magic (RoM)



$$\mathcal{R}(\rho) = \min_{\rho^+,\rho^- \in \mathrm{STAB}_n} \{2a+1 \mid \rho = (a+1)\rho^+ - a\rho^-, a \geq 0\}$$

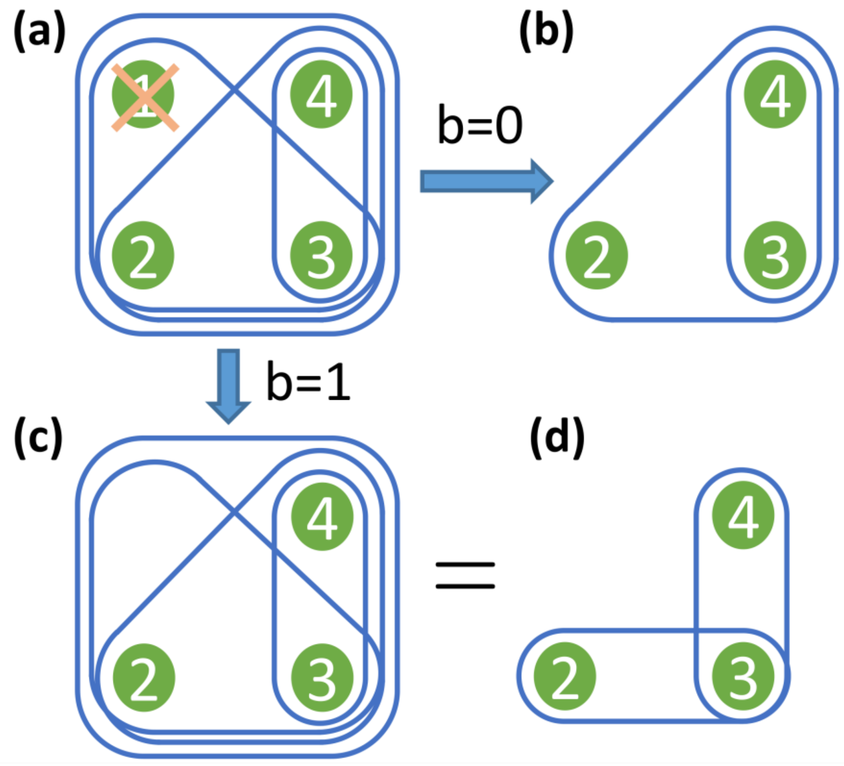
1. $\mathcal{R}(\cdot)$ is faithful, i.e., $\mathcal{R}(\sigma) = 1$ iff $\sigma \in \mathrm{STAB}$;
2. For all trace-preserving stabilizer channels $\mathcal{E}$, we have $\mathcal{R}(\mathcal{E}(\rho)) \leq \mathcal{R}(\rho)$;
3. $\mathcal{R}(\sigma \otimes \rho) = \mathcal{R}(\rho)$ for $\sigma \in \mathrm{STAB}$;
4. For a set of states $\{\rho_k\}$ and a set of real numbers $\{p_k\}$ satisfying $\sum_k p_k = 1$, the *convexity* of $\mathcal{R}(\cdot)$ implies that $\mathcal{R}(\sum_k p_k \rho_k) \leq \sum_k |p_k| \mathcal{R}(\rho_k)$;
5. Classical simulation overhead: $\mathcal{R}(\rho)^2$.

Howard M, Campbell E. Application of a resource theory for magic states to fault-tolerant quantum computing[J]. PRL, 2017, 118(9): 090501.

Noise robustness and threshold of magic

- Consider the n-qubit independent depolarizing noise

$$\mathcal{E}_\lambda^{\otimes n} = ((1-\lambda)\mathcal{I} + \lambda\mathcal{G})^{\otimes n}, \text{ where } \mathcal{G}(\sigma) = \mathrm{Tr}(\sigma)\mathbb{I}_2/2.$$



For an n-qubit hypergraph state $\Psi := |\Psi\rangle\langle\Psi|$, and a subset of qubits $I \subset [n]$, consider tracing out the qubits in I. We have

$$\mathrm{Tr}_I(\Psi) = \frac{1}{2^{|I|}} \sum_{\mathbf{b}\in\mathbb{Z}_2^{|I|}} \Psi^{(I,\mathbf{b})}, \tag{1}$$

where $\Psi^{(I,\mathbf{b})}$ are $(n-|I|)$-qubit hypergraph states obtained by removing vertices and edges from Ψ.

Noise robustness and threshold of magic

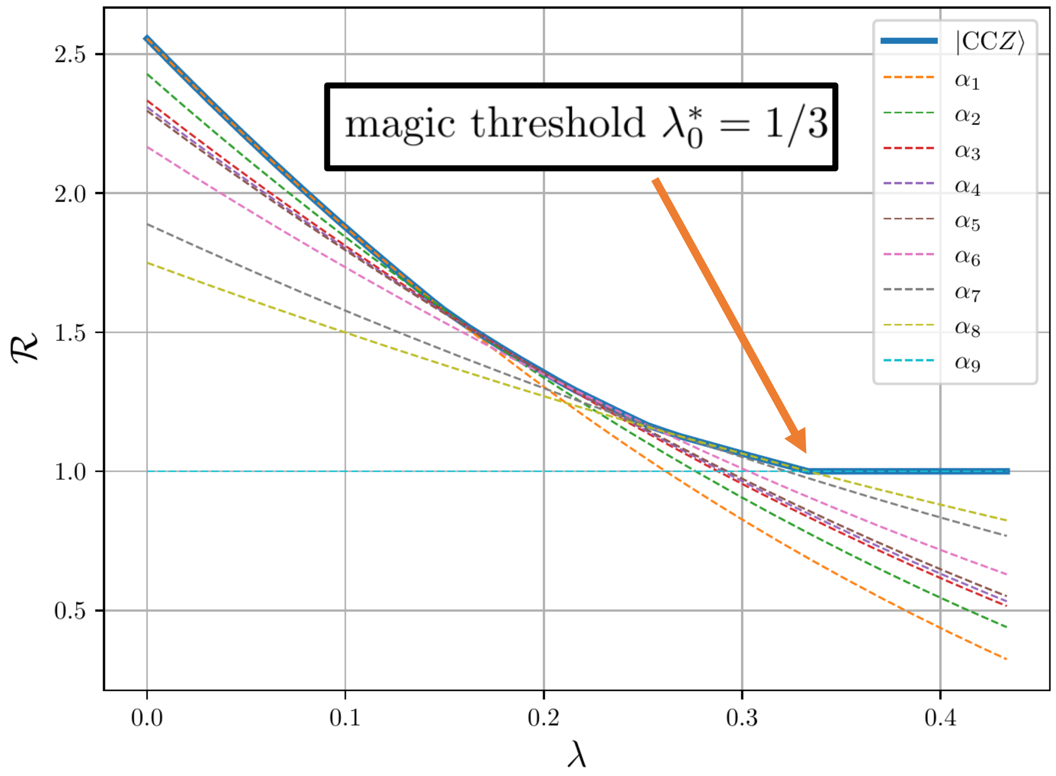
- The decay profile of magic: $\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\rho)\right)$
- The magic noise threshold above which the magic is eliminated.

Definition 1. For a state ρ and $\epsilon \geq 0$, we call $\lambda_\epsilon^*(\rho) := \inf_{\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\rho)\right) \leq 1+\epsilon} \lambda$ the *ϵ-magic noise threshold*. For a family of states $\{\rho_n\}$ where ρ_n is an n-qubit state, if for a fixed $\epsilon \geq 0$ we have $\liminf_{n\to\infty} \lambda_\epsilon^*(\rho_n) > 0$, we say $\{\rho_n\}$ has a *non-vanishing magic threshold*.

Noise robustness and threshold of magic

- Warm up example: $\Phi = |\mathrm{CCZ}\rangle\langle\mathrm{CCZ}|$, where $|\mathrm{CCZ}\rangle \coloneqq \mathrm{CCZ}|+^3\rangle$

$\mathcal{R}(\mathcal{E}_\lambda^{\otimes n}(\Phi)) = \max_{1\le j\le 9}\alpha_j(\lambda)$, where $\alpha_j(\lambda) = \mathrm{Tr}\left(\mathcal{E}_\lambda^{\otimes 3}(\Phi)A_j\right)$ are polynomials in λ with degree at most 3, with A_j some Hermitian matrices.



- Computing the decay profile for $n > 5$ is hard in general.

Noise robustness and threshold of magic

- Key technique: upper and lower bounds for noisy RoM.
- Upper bound: by convexity of $\mathcal{R}(\cdot)$

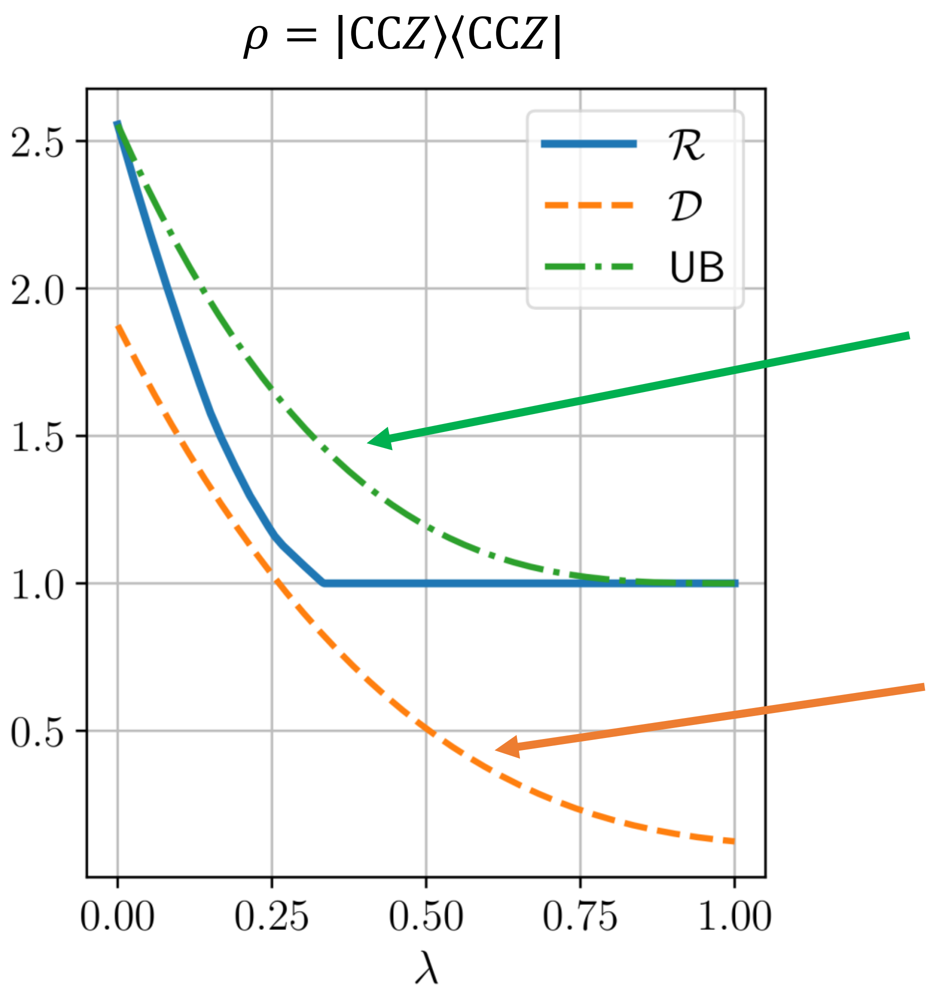
$$\mathcal{R}\left(\mathcal{E}_{\lambda}^{\otimes n}(\rho)\right) \leq \sum_{I \subset [n]} (1-\lambda)^{n-|I|} \lambda^{|I|} \mathcal{R}\left(\mathrm{Tr}_I(\rho)\right)$$

- Lower bound: by $\mathcal{R}(\rho) \geq \mathcal{D}(\rho) := \frac{1}{2^n} \sum_{P \in \mathrm{P}_n^+} |\mathrm{Tr}(P\rho)|$

$$\mathcal{R}\left(\mathcal{E}_{\lambda}^{\otimes n}(\rho)\right) \geq \frac{1}{2^n} \sum_{P \in \mathrm{P}_n^+} (1-\lambda)^{\mathrm{wt}(P)} |\mathrm{Tr}(P\rho)|$$

Howard M, Campbell E. Application of a resource theory for magic states to fault-tolerant quantum computing[J]. PRLs, 2017, 118(9): 090501.
Leone L, Oliviero S F E, Hamma A. Stabilizer rényi entropy[J]. PRL, 2022, 128(5): 050402.

Noise robustness and threshold of magic



$$\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\rho)\right) \leq \sum_{I\subset[n]} (1-\lambda)^{n-|I|}\lambda^{|I|}\mathcal{R}\left(\mathrm{Tr}_I(\rho)\right)$$

$$\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\rho)\right) \geq \frac{1}{2^n}\sum_{P\in \mathrm{P}_n^+} (1-\lambda)^{\mathrm{wt}(P)}\left|\mathrm{Tr}(P\rho)\right|$$

Result 1. High-degree edges are fragile

- Hypergraph state $\Phi_n = |C^{n-1}Z\rangle\langle C^{n-1}Z|$, a single edge containing all qubits.
  - we can show that $\mathcal{R}(|C^{n-1}Z\rangle) < 5$, for all n
  - magic spreads more extensively, more exposed to noise as n increases

$$2\left(1 - 3\lambda/4\right)^n \lesssim \mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}\left(\Phi_n\right)\right) \leq 1 + 4\left(1 - \lambda/2\right)^n$$

Theorem 1. *For a fixed noise rate* $0 < \lambda < 1$ *and any* $\epsilon > 0$, *the family* $\{|C^{n-1}Z\rangle\}$ *has magic threshold* $\lambda_\epsilon^* = \Theta(n^{-1})$.

Result 1. High-degree edges are fragile

Theorem 1. *For a fixed noise rate $0 < \lambda < 1$ and any $\epsilon > 0$, the family $\{|\mathrm{C}^{n-1}Z\rangle\}$ has magic threshold $\lambda_\epsilon^* = \Theta(n^{-1})$.*

- $\mathrm{C}^{n-1}Z$ variants of magic state distillation and injection?

At least $\Omega((1-\lambda/2)^{-n})$ copies of $\mathcal{E}_\lambda^{\otimes n}(\Phi_n)$ are required to achieve constant RoM, and therefore, to distill Φ_n with constant success probability.

Result 1. High-degree edges are fragile

- Furthermore, in an arbitrary hypergraph state Ψ, the magic contributed by high-degree edges is fragile to noise.

Theorem 2. *Let Ψ be an n-qubit hypergraph state. Consider adding K edges $e_1, \cdots, e_K \subset [n]$ to Ψ to obtain a new state Φ. Then $\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\Phi)\right)$ is upper bounded by*

$$\mathcal{R}\left(\mathcal{E}_\lambda^{\otimes n}(\Psi)\right) + C_\Psi \sum_{\emptyset \neq J \subset [K]} (5^{|J|}+1)\Big(1-\frac{\lambda}{2}\Big)^{|\cup_{j\in J} e_j|},$$

where $C_\Psi := \max_{I\subset[n], \mathbf{s}\in\mathbb{Z}_2^{|I|}} \mathcal{R}\big(\Psi^{(I,\mathbf{s})}\big)$ is a constant depending on Ψ.

Result 2. Local magic and magic threshold

For an n-qubit state ρ, define its maximum K-local RoM by

$$\mathcal{M}_K(\rho) := \max_{J \subset [n], |J| = K} \mathcal{R}\left(\mathrm{Tr}_{\overline{J}}(\rho)\right),$$

where $\overline{J} := [n] - J$.

For a family of states $\{\rho_n\}$, if there exists a constant K such that $\liminf_{n\to\infty} \mathcal{M}_K(\rho_n) > 1$ (or $\lim_{n\to\infty} \mathcal{M}_K(\rho_n) = 1$), we say $\{\rho_n\}$ has *non-vanishing (or vanishing) local magic*.

Result 2. Local magic and magic threshold

Proposition 3. *Suppose $\{\rho_n\}$ is a family of states with non-vanishing local magic, then it has a non-vanishing magic threshold.*

Let J_n ($|J_n| = K$) denote the K qubits on which ρ_n has maximum RoM. Since partial trace commutes with local noises, to ensure $\mathcal{E}_\lambda^{\otimes n}(\rho_n) \in \mathrm{STAB}_n$, we must have $\mathcal{E}_\lambda^{\otimes K}\left(\mathrm{Tr}_{\overline{J_n}}(\rho_n)\right) \in \mathrm{STAB}_K$. Since the K-qubit state $\mathrm{Tr}_{\overline{J_n}}(\rho_n)$ has RoM $\geq$ constant for all sufficiently large n, it can withstand constant rate of noises before falling into STAB_K.

Result 2. Local magic and magic threshold

Theorem 4. *There exist efficiently preparable families of many-body states with vanishing local magic yet a non-vanishing magic threshold.*

For example, consider the family of 3-complete hypergraph states $\{\Gamma_n\}$, where Γ_n corresponds to the hypergraph with n vertices and all possible edges of degree 3. Evidently, $\{\Gamma_n\}$ has gate complexity $\mathcal{O}(n^3)$.

Result 2. Local magic and magic threshold

Theorem 4. *There exist efficiently preparable families of many-body states with vanishing local magic yet a non-vanishing magic threshold.*

For any constant K, the reduced density matrix of Γ_n on K qubits is a convex combination of four $(n-K)$-qubit hypergraph states, with all four coefficients approaching 1/4 as $n \to \infty$, which can be proved to form a stabilizer state. Specifically, in Proposition 13, we show that $\mathcal{M}_K(\Gamma_n) \leq 1 + 2^{1+\frac{3}{2}K-\frac{n}{2}}$, which tends to 1 for all K as $n \to \infty$, indicating that $\{\Gamma_n\}$ has vanishing local magic.

Result 2. Local magic and magic threshold

Theorem 4. *There exist efficiently preparable families of many-body states with vanishing local magic yet a non-vanishing magic threshold.*

By upper and lower bounds for $\mathcal{R}\left(\mathcal{E}_{\lambda}^{\otimes n}(\Gamma_n)\right)$, the magic threshold for $\{\Gamma_n\}$ satisfies $0.39 \leq \lambda_{\epsilon}^{*} \leq 0.78$, demonstrating a non-vanishing threshold.

Result 2. Local magic and magic threshold

- Insights into interplay between magic and entanglement:

Γ_n has a vanishing amount of magic even when restricted to $\Theta(n)$ qubits: dividing Γ_n evenly among 4 parties results in a vanishing amount of magic for each party, as

$$\mathcal{M}_{n/4}(\Gamma_n) \leq 1 + 2^{1-n/8} \to 1.$$

However, the total magic remains substantial, as $\mathcal{R}(\Gamma_n) \geq \mathcal{D}(\Gamma_n) = \Theta(2^{n/2})$.

Result 2. Local magic and magic threshold

- Insights into interplay between magic and entanglement:

> The "non-local magic" in these states is embedded in entanglement and can withstand a constant rate of noise. We anticipate that such states may find interesting applications through "magic hiding" or "magic secret sharing".

Noise robustness and threshold of magic

Hypergraph state Ψ	Edge degree	Threshold	$\mathcal{R}(\mathcal{E}_\lambda^{\otimes n}(\Psi))$ upper bound	$\mathcal{R}(\mathcal{E}_\lambda^{\otimes n}(\Psi))$ lower bound	Local magic
$\lvert \mathrm{CC}Z\rangle$	3	$\lambda_0^* = 1/3$	$\mathcal{R}(\mathcal{E}_\lambda^{\otimes n}(\Psi)) = \max_{1\leq j\leq 9} \alpha_j(\lambda)$ [Fig. 3]		$23/9$
Union Jack lattice		$\lambda_0^* \geq \text{constant} > 0$	$\mathcal{O}\left((2-\lambda)^n\right)$	–	1.0078
3-complete hypergraph		$0.39 \leq \lambda_\epsilon^* \leq 0.78$	$1 + 2^{n+1}\left(1-(1-2^{-3/2})\lambda\right)^n$	$\approx 2^{(n-3)/2}(1-3\lambda/4)^n$	$1+\mathcal{O}(2^{-n/2})$
4-complete hypergraph	4	$\lambda_0^* \geq \text{constant} > 0$	$\mathcal{O}\left((2-\lambda)^n\right)$	$\mathcal{D}\left(\mathcal{E}_\lambda^{\otimes n}(\Psi)\right)$ [Fig. 5]	$1.25 + o(1)$
High-degree hypergraphs	$\geq m$	$\lambda_\epsilon^* = \mathcal{O}(1/m)$	$1 + \text{poly}(m)(1-\lambda/2)^m$	–	$1 + o(1)$
$\lvert \mathrm{C}^{n-1}Z\rangle$	n	$\lambda_\epsilon^* = \Theta(1/n)$	$1 + 4(1-\lambda/2)^n$	$\approx 2\left(1-3\lambda/4\right)^n$	$1+\Theta\left((1/2)^n\right)$
Qudit $\lvert \mathrm{C}^{n-1}Z\rangle$	n	$\lambda_0^* \geq 0.42\ (d=3)$	$1 + 4M_d(d-1)^n\left(1-\frac{d-1}{d}\lambda\right)^n$	$1 + 2\,\text{sn}(\mathcal{E}_\lambda^{\otimes n}(\Psi))$ [Fig. 6]	$1+\mathcal{O}\left(\left(\frac{d-1}{d}\right)^n\right)$

Take home messages

- Decay RoM of hypergraph states under independent noise.
- High-degree edges are fragile
- 3-complete hypergraph state
  - vanishing local magic (even on $\Theta(n)$ qubits)
  - non-vanishing magic threshold

Outlook

- Identify the most robust hypergraph states.
- The necessary and sufficient conditions for a family of states to have a non-vanishing magic threshold.
- The noise robustness of MBQC power.

Thank you for your attention!

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